Transposition Solutions to Backward Stochastic Evolution Equations in Infinite Dimensions and Their Applications

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Outline

1 Introduction

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, on which a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined so that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $\{W(t)\}_{t\geq 0}$.

1 Introduction

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- We define

$$\mathcal{U}[0,T] \stackrel{\triangle}{=} \{ u : [0,T] \times \Omega \to U \mid u \text{ is } \mathbb{F}\text{-adapted} \}.$$

- We consider the following stochastic controlled system:

$$\begin{cases} dx(t)=b(t,x(t),u(t))dt+\sigma(t,x(t),u(t))dW(t), t\in[0,T], \\ x(0)=x_0, \end{cases} \tag{1}$$

with the cost functional

$$\mathcal{J}(u(\cdot))=\mathbb{E}\Big\{\int_0^T f(t,x(t),u(t))dt+h(x(T))\Big\}.$$

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- **Problem (S).** Minimize $\mathcal{J}(\cdot)$ over $\mathcal{U}[0,T]$.
- Any $\overline{u}(\cdot)\in\mathcal{U}[0,T]$ satisfying

$$J(\overline{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)),$$

is called an *optimal control*, the corresponding $\overline{x}(\cdot)\equiv x(\cdot\,;\overline{u}(\cdot))$ and $(\overline{x}(\cdot),\overline{u}(\cdot))$ are called an *optimal state process/trajectory* and an *optimal pair*, respectively.

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- A systematic solution is given by E. Pardoux and S. Peng in 1990, and a general maximum principle was given by S. Peng in 1990.

- Fix any $u(\cdot) \in \mathcal{U}[0, T]$ and $\varepsilon > 0$. Define

$$u^{\varepsilon}(t) = \begin{cases} \overline{u}(t), & t \in [0, T] \setminus E_{\varepsilon}, \\ u(t), & t \in E_{\varepsilon}, \end{cases}$$

where $E_{\varepsilon} \subseteq [0, T]$ is a measurable set with $|E_{\varepsilon}| = \varepsilon$.

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where $E_{\varepsilon} \subseteq [0,T]$ is a measurable set with $|E_{\varepsilon}| = \varepsilon$.

- Let $(x^{\varepsilon}(\cdot), u^{\varepsilon}(\cdot))$ satisfy the following:

$$\begin{cases} dx^{\varepsilon}(t) = b(t, x^{\varepsilon}(t), u^{\varepsilon}(t))dt + \sigma(t, x^{\varepsilon}(t), u^{\varepsilon}(t))dW(t), \ t \in [0,T], \\ x^{\varepsilon}(0) = x_0. \end{cases}$$

- One can show that

$$
\begin{aligned}
&\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\overline{u}(\cdot))\\
&= \mathbb{E}\big\langle h_x(\overline{x}(T)), y^\varepsilon(T)+z^\varepsilon(T)\big\rangle + \frac{1}{2}\mathbb{E}\big\langle h_{xx}(\overline{x}(T))y^\varepsilon(T), y^\varepsilon(T)\big\rangle\\
&\quad +\mathbb{E}\int_0^T \Big\{\big\langle f_x(t), y^\varepsilon(t)+z^\varepsilon(t)\big\rangle + \frac{1}{2}\big\langle f_{xx}(t)y^\varepsilon(t), y^\varepsilon(t)\big\rangle\\
&\qquad\qquad +\delta f(t)\chi_{E_\varepsilon}(t)\Big\}dt + o(\varepsilon).
\end{aligned}
$$

- $y^{\varepsilon}(\cdot)$ solves:

$$\begin{cases} dy^{\varepsilon}(t) = b_x(t)y^{\varepsilon}(t)dt + \left\{\sigma_x(t)y^{\varepsilon}(t) + \delta\sigma(t)\chi_{E_{\varepsilon}}(t)\right\}dW(t), \\ \hfill t \in [0, T], \\ y^{\varepsilon}(0) = 0. \end{cases}$$

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$$\begin{cases} dz^{\varepsilon}(t) = \Big\{b_x(t)z^{\varepsilon}(t) + \delta b(t)\chi_{E_\varepsilon}(t) + \dfrac{1}{2}y^{\varepsilon}(t)b_{xx}(t)y^{\varepsilon}(t)\Big\}dt \\ \qquad + \Big\{\sigma_x(t)z^{\varepsilon}(t) + \delta\sigma_x(t)y^{\varepsilon}(t)\chi_{E_\varepsilon}(t) \\ \qquad\qquad + \dfrac{1}{2}y^{\varepsilon}(t)\sigma_{xx}(t)y^{\varepsilon}(t)\Big\}dW(t), \quad t \in [0,T], \\ z^{\varepsilon}(0) = 0. \end{cases}$$

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- $$\begin{cases} dp(t) = -\Big\{ b_x(t,\overline{x}(t),\overline{u}(t))^* p(t) + \sigma_x(t,\overline{x}(t),\overline{u}(t))^* q(t) \\ \qquad\qquad -f_x(t,\overline{x}(t),\overline{u}(t))\Big\} dt + q(t)dW(t), \quad t\in[0,T], \\ p(T) = -h_x(\overline{x}(T)). \end{cases} \tag{2}$$

- One has to introduce another variable to reflect the uncertainty in the system. This is done by introducing an additional adjoint equation as follows:

$$
\begin{cases}
dP(t) = -\Big\{ b_x(t,\overline{x}(t),\overline{u}(t))^\top P(t) + P(t) b_x(t,\overline{x}(t),\overline{u}(t)) \\
\qquad\qquad + \sigma_x(t,\overline{x}(t),\overline{u}(t))^\top P(t)\sigma_x(t,\overline{x}(t),\overline{u}(t)) \\
\qquad\qquad + \sigma_x(t,\overline{x}(t),\overline{u}(t))^\top Q(t) + Q(t)\sigma_x(t,\overline{x}(t),\overline{u}(t)) \\
\qquad\qquad + H_{xx}(t,\overline{x}(t),\overline{u}(t),p(t),q(t)) \Big\} dt + Q(t)dW(t), \\
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P(T) = -h_{xx}(\overline{x}(T)),
\end{cases}
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$$

- Here the *Hamiltonian* H is defined by

$$
\begin{aligned}
H(t,x,u,p,q) &= p^\top b(t,x,u) + q^\top \sigma(t,x,u) - f(t,x,u), \\
&\qquad (t,x,u,p,q) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n,
\end{aligned}
\tag{4}
$$

and $(p(\cdot), q(\cdot))$ is the solution to (2). In the above (3), the unknown is again a pair of processes $(P(\cdot), Q(\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathcal{S}^n) \times L^2_{\mathbb{F}}(0,T;\mathcal{S}^n)$.

- Let

$$
\begin{aligned}
&\mathcal{H}(t,x,u)\\
&\triangleq H(t,x,u,p(t),q(t))-\frac{1}{2}\sigma(t,\overline{x}(t),\overline{u}(t))^\top P(t)\sigma(t,\overline{x}(t),\overline{u}(t))\\
&+\frac{1}{2}\mathrm{tr}\Big\{\big[\sigma(t,x,u)-\sigma(t,\overline{x}(t),\overline{u}(t))\big]^\top P(t)\\
&\qquad\qquad\cdot\big[\sigma(t,x,u)-\sigma(t,\overline{x}(t),\overline{u}(t))\big]\Big\}.
\end{aligned}
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$$\begin{aligned}&\mathcal{H}(t,x,u)\\ \stackrel{\triangle}{=}&H(t,x,u,p(t),q(t))-\tfrac{1}{2}\sigma(t,\bar{x}(t),\bar{u}(t))^\top P(t)\sigma(t,\bar{x}(t),\bar{u}(t))\\ &+\frac{1}{2}\mathrm{tr}\Big\{\big[\sigma(t,x,u)-\sigma(t,\bar{x}(t),\bar{u}(t))\big]^\top P(t)\\ &\qquad\qquad\cdot\big[\sigma(t,x,u)-\sigma(t,\bar{x}(t),\bar{u}(t))\big]\Big\}.\end{aligned}$$

- **Theorem 3.2.** (Stochastic Maximum Principle) *Let b, σ, f and h are smooth enough. Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be an optimal pair of Problem (S). Then*

$$\mathcal{H}(t,\bar{x}(t),\bar{u}(t))=\max_{u\in U}\mathcal{H}(t,\bar{x}(t),u),\quad \text{a.e.}\, t\in[0,T],\quad \mathbb{P}\text{-a.s.}$$

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- It is for controlled stochastic ODEs. Can it be generalized to controlled stochastic PDEs?

- To achieve the above goal, we meet two main problems.

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- To obtain the maximum principle for general filtration $\mathbb{F}$, one need to solve BSDEs on general filtration. In this case, the Martingale Representation Theorem, which is a key point to obtain the well-posedness of BSDEs by Pardoux and Peng, does not hold.

- To achieve the above goal, we meet two main problems.
- To obtain the maximum principle for general filtration $\mathbb{F}$, one need to solve BSDEs on general filtration. In this case, the Martingale Representation Theorem, which is a key point to obtain the well-posedness of BSDEs by Pardoux and Peng, does not hold.
- The work by N. El Karoui and S.-J. Huang (1997) shows that one needs to introduce an extra corrected term to (8), and therefore, it is even more difficult to "compute" the above $Y(\cdot)$.

- Recently, by replacing $Y(t)dw(t)$ in (8) by $dM(t)$ (with $M(\cdot)$ being a square-integrable martingale), G. Liang T. Lyons and Z. Qian (2008) developed another approach for the well-posedness of BSDEs with the general filtration.

- Recently, by replacing $Y(t)dw(t)$ in (8) by $dM(t)$ (with $M(\cdot)$ being a square-integrable martingale), G. Liang T. Lyons and Z. Qian (2008) developed another approach for the well-posedness of BSDEs with the general filtration.

- The advantage of this approach is that martingale representation theorem is not required, either. But the cost is that the corrected term $Y(\cdot)$ in (8) is suppressed. Note that this term plays a crucial role in some problems, say the Pontryagin-type maximum principle for general stochastic optimal control problems. Also, the comparison theorem is not clear in this setting because the usual duality analysis is not available.

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- It does not make any really trouble once we know how to solve vector backward SDEs, thanks to that a $n \times n$-matrix can be regarded as a vector in $\mathbb{R}^{n \times n}$.

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- One has to face a real challenge in the study of maximum principle for controlled SPDEs. Indeed, in the infinite dimensional setting, we should introduced an operator valued BSDE.
- Although the set of all bounded linear operators (with the operator topology) is still a Banach space, it is not UMD.
- There exists no such a stochastic integration/evolution equation theory in general Banach spaces that can be employed to treat such equations.

- To overcome the two difficulties mentioned above, we give a weaker but reasonable definition for the solution to vector valued BSDE and the operator valued BSDE, motivated by the transposition solution to partial differential equations with non-homogeneous boundary condition, and prove the corresponding well-posedness result.

2. The classical transposition method in PDEs

- We now recall the main idea in the classical transposition method to solve wave equation with non-homogeneous Dirichlet boundary conditions.

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- We now recall the main idea in the classical transposition method to solve wave equation with non-homogeneous Dirichlet boundary conditions.
- Consider the following wave equation:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q \equiv (0,T) \times G, \\ y = u & \text{on } \Sigma \equiv (0,T) \times \Gamma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \tag{5}$$

where $T > 0$, G is a nonempty open bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}$) with C^2 boundary Γ, $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$ and $u \in L^2((0,T) \times \Gamma)$.

- When $u \equiv 0$, one can use the standard Semigroup Theory to show the well-posedness of (5).

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- When $u \not\equiv 0$, one needs to use the transposition method. For this purpose, for any $f \in L^1(0,T;L^2(\Omega))$ and $g \in L^1(0,T;H_0^1(\Omega))$, consider the following adjoint problem of (5):

$$\begin{cases} \zeta_{tt} - \Delta\zeta = f + g_t, & \text{in } Q, \\ \zeta = 0, & \text{on } \Sigma, \\ \zeta(T) = \zeta_t(T) = 0, & \text{in } G. \end{cases} \tag{6}$$

It is easy to show that the equation (6) admits a unique solution $\zeta \in C([0,T];H_0^1(G)) \bigcap C^1([0,T];L^2(G))$, which enjoys the regularity $\frac{\partial\zeta}{\partial\nu} \in L^2(\Sigma)$.

- In order to give a reasonable definition for the solution to the non-homogenous boundary problem (5) by the transposition method, we consider first the case when y is sufficiently smooth.

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- Assume $g \in C_0^\infty([0,T]; H_0^1(G))$ and that $y \in H^2(Q)$ satisfies (5). Then

$$\begin{aligned}&\int_Q fy\,dxdt - \int_Q g y_t\,dxdt\\&= \int_G \zeta(0) y_1\,dx - \int_G \zeta_t(0) y_0\,dx - \int_\Sigma \frac{\partial \zeta}{\partial \nu} u\,d\Sigma.\end{aligned} \tag{7}$$

- Note that (7) still makes sense even if the regularity of y is relaxed as $y \in C([0,T]; L^2(G)) \bigcap C^1([0,T]; H^{-1}(G))$. Because of this, one introduces the following:

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- **Definition 1**. We call $y \in C([0,T];L^2(G)) \bigcap C^1([0,T];H^{-1}(G))$ a solution to (5), in the sense of transposition, if $y(0) = y_0$, $y_t(0) = y_1$, and for any $f \in L^1(0,T;L^2(G))$ and any $g \in L^1(0,T;H_0^1(G))$, it holds that

$$\begin{aligned}&\int_Q fydxdt - \int_0^T \langle g, y_t\rangle_{H_0^1(G),H^{-1}(G)}dt\\&= \langle \zeta(0), y_1\rangle_{H_0^1(G),H^{-1}(G)} + \int_\Omega \zeta_t(0)y_0dx - \int_\Sigma \frac{\partial \zeta}{\partial \nu}ud\Sigma,\end{aligned}$$

where ζ is the unique solution to (6).

- One can show the well-posedness of (5) in the sense of transposition. The principle idea of this method is to interpret the solution to one forward wave equation with non-homogeneous Dirichlet boundary conditions in terms of another backward wave equation with non-homogeneous source terms.

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- The transposition method is a variant of duality method. Like a mirror, it provides a way to see something which is not easy to be detected directly.
- We shall use this idea to interpret BSDEs/BSEEs in terms of SDEs/SEEs.

3. Transposition solution to BSDEs

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, on which a one dimensional standard Brownian motion $\{W(t)\}_{t\in[0,T]}$ is defined.

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- Consider the following semilinear BSDE:

$$\begin{cases} dy(t) = f(t, y(t), Y(t))dt + Y(t)dW(t) \quad \text{in } [0, T], \\ y(T) = y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n), \end{cases} \tag{8}$$

where $f(\cdot,\cdot,\cdot)$ satisfies the usual Lipschitz condition and $f(\cdot, 0, 0) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}^n))$.

- Similar to the transposition method for non-homogeneous boundary value problems, for fixed $t \in [0, T]$, we start from the following linear (forward) stochastic differential equation

$$\begin{cases} dz(\tau) = u(\tau)d\tau + v(\tau)dW(\tau), \quad \tau \in (t, T], \\ z(t) = \eta. \end{cases} \tag{9}$$

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- It is clear that, for given $u(\cdot) \in L^1_{\mathbb{F}}(t, T; L^2(\Omega; \mathbb{R}^n))$, $v(\cdot) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$, equation (9) admits a unique solution $z(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$.

- Now, if equation (8) admits a classical solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T; \mathbb{R}^n)$, then, applying Itô's formula to $\langle z(t), y(t)\rangle$, it follows

$$
\begin{aligned}
&\mathbb{E}\langle z(T), y_T\rangle - \mathbb{E}\langle \eta, y(t)\rangle \\
&= \mathbb{E}\int_t^T \langle z(\tau), f(\tau, y(\tau), Y(\tau))\rangle d\tau \\
&\quad + \mathbb{E}\int_t^T \langle u(\tau), y(\tau)\rangle d\tau + \mathbb{E}\int_t^T \langle v(\tau), Y(\tau)\rangle d\tau.
\end{aligned}
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- This inspires us to introduce the following new notion for the solution to the equation (8).

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- **Definition 2**. We call $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^2(\Omega;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ a transposition solution to the equation (8) if for any $t \in [0,T]$, $u(\cdot) \in L^1_{\mathbb{F}}(t,T;L^2(\Omega;\mathbb{R}^n))$, $v(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, it holds that

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\end{aligned} \tag{10}$$

- Clearly, any transposition solution to the equation (8) coincides with its classical solution whenever the filtration $\mathbb{F}$ is natural one generated by $W(\cdot)$.

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- **Theorem 1**. (Q. Lü and X. Zhang) For any given $y_T \in L^2_{\mathcal{F}_T}(\Omega)$, the equation (8) admits a unique transposition solution $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^2(\Omega;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Furthermore, there is a constant $C>0$, depending only on K and T, such that

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([0,T];L^2(\Omega;\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)} \\
&\leq C\left[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right].
\end{aligned}
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\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([0,T];L^2(\Omega;\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)}\\
&\leq C\left[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right].
\end{aligned}
\tag{11}
$$

- Our method does not need the martingale representation theorem.

4. Well-posedness of vector-valued BSEEs

- Consider the following vector-valued backward stochastic differential equation:

$$\begin{cases} dy = -A^* y dt + f(t, y, Y)dt + Y dW(t) \quad \text{in } [0, T), \\ y(T) = y_T. \end{cases} \tag{12}$$

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- To define the transposition solution to (12), we introduce the following forward stochastic differential equation:

$$\begin{cases} dz = (Az + v_1)dt + v_2 dW(t) \quad \text{in } (t, T], \\ z(t) = \eta. \end{cases} \tag{13}$$

Here $v_1(\cdot) \in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $v_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, and $\frac{1}{p} + \frac{1}{q} = 1$.

- **Definition 3**. We call $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0,T;\ L^p(\Omega; H))$ a transposition solution to (12) if for any $t \in [0,T]$, $v_1(\cdot) \in L^1_{\mathbb{F}}(t,T; L^q(\Omega; H))$, $v_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^q(\Omega; H))$ and $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle z(T), y_T\big\rangle_H - \mathbb{E}\int_t^T \big\langle z(s), f(s, y(s), Y(s))\big\rangle_H \\
&= \mathbb{E}\big\langle \eta, y(t)\big\rangle_H + \mathbb{E}\int_t^T \big\langle v_1(s), y(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle v_2(s), Y(s)\big\rangle_H ds.
\end{aligned}
$$

- **Theorem 3**. (Q. Lü and X. Zhang) *Let H be a Hilbert space. For any $y_T \in L^p_{\mathcal{F}_T}(\Omega; H)$, and any $f(\cdot,\cdot,\cdot) : [0,T]\times H\times H \to H$, the equation (12) admits one and only one unique transposition solution*

$$(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega;H)) \times L^2_{\mathbb{F}}(0,T; L^p(\Omega;H)).$$

Furthermore, there is a constant C such that

$$\begin{aligned} &|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([t,T];L^p(\Omega;H))\times L^2_{\mathbb{F}}(t,T;L^p(\Omega;H))} \\ &\leq C\left[|f(\cdot,0,0)|_{L^1_{\mathbb{F}}(t,T;L^p(\Omega;H))} + |y_T|_{L^p_{\mathcal{F}_T}(\Omega;H)}\right], \\ &\quad \forall\, t\in[0,T]. \end{aligned} \tag{14}$$

5. *Well-posedness of operator-valued BSEEs*

- Consider the following operator-valued backward stochastic evolution equation:

$$\begin{cases} dP = -(A^* + J^*(t))Pdt - P(A + J(t))dt - K^*PKdt \\ \qquad\qquad -(K^*Q + QK)dt + Fdt + QdW(t) \quad \text{in } [0,T), \\ P(T) = P_T. \end{cases} \tag{15}$$

Here $F \in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$, and $J, K \in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))$.

- In order to define the transposition solution to the equation (15), we introduce the following two stochastic differential equation:

$$\begin{cases} dx_1 = (A+J)x_1 ds + u_1 ds + K x_1 dW(t) + v_1 dW(t) \quad \text{in } (t, T], \\ x_1(t) = \xi_1, \end{cases}$$

$$\begin{cases} dx_2 = (A+J)x_2 ds + u_2 ds + K x_2 dW(t) + v_2 dW(t) \quad \text{in } (t, T], \\ x_2(t) = \xi_2. \end{cases}$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1, v_2 \in L^4_{\mathbb{F}}(t, T; L^4(\Omega; H))$.

Definition 4. We call $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F},w}([0,T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathbb{F},w}(0,T; L^2(\Omega; \mathcal{L}(H)))$ a transposition solution to (15) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds \\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle Q(s)v_1(s), x_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q(s)x_1(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

- Denote by $\mathcal{L}_2(H)$ the set of the Hilbert-Schmidt operators on H.

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- **Theorem 4**. (Q. Lü and X. Zhang) *Assume that H is a separable Hilbert space and $L^p_{\mathcal{F}_T}(\Omega)$ $(1 \leq p < \infty)$ is a separable Banach space. Then, for any $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}_2(H))$, $F \in L^1_{\mathbb{F}}(0,T; L^2(\Omega; \mathcal{L}_2(H)))$ and $J, K \in L^4_{\mathbb{F}}(0,T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (15) admits one and only one transposition solution (P,Q) with the regularity $\big(P(\cdot), Q(\cdot)\big) \in D_{\mathbb{F}}([0,T]; L^2(\Omega; \mathcal{L}_2(H))) \times L^2_{\mathbb{F}}(0,T; \mathcal{L}_2(H))$. Furthermore,*

$$
\begin{aligned}
&|(P,Q)|_{D_{\mathbb{F}}([0,T];L^2(\Omega;\mathcal{L}_2(H)))\times L^2_{\mathbb{F}}(0,T;\mathcal{L}_2(H))} \\
&\leq C\Big[|F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}_2(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}_2(H))}\Big].
\end{aligned}
\tag{16}
$$

- Theorems 4 indicates that, in some sense, the transposition solution introduced in Definition 4 is a reasonable notion for the solution to (15).

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- Unfortunately, we are unable to prove the existence of transposition solution to (15) in the general case.

- We shall introduced below a weaker version of solution, i.e., the relaxed transposition solution (to (15)), which looks awkward but it suffices to establish the desired Pontryagin-type stochastic maximum principle for optimal control of controlled stochastic evolution equations.

- **Definition 5**. We call $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$ a relaxed transposition solution to (15) if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t, T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds \\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds \\
&+\mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds \\
&+\mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds \\
&+\mathbb{E}\int_t^T \big\langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s)\big\rangle_H ds \\
&+\mathbb{E}\int_t^T \big\langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

- It is easy to see that, if $\big(P(\cdot), Q(\cdot)\big)$ is a transposition solution to (15), then one can find a relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$ to the same equation (from $\big(P(\cdot), Q(\cdot)\big)$). Indeed, they are related by

$$
\begin{aligned}
Q(s)x_1(s) &= Q^{(t)}(\xi_1, u_1, v_1)(s),\\
Q(s)^* x_2(s) &= \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s).
\end{aligned}
$$

  This means that, we know only the action of $Q(s)$ (or $Q(s)^*$) on the solution processes $x_1(s)$ (or $x_2(s)$).

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  This means that, we know only the action of $Q(s)$ (or $Q(s)^*$) on the solution processes $x_1(s)$ (or $x_2(s)$).

- However, it is unclear how to obtain a transposition solution $\big(P(\cdot), Q(\cdot)\big)$ to (15) by means of its relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$. It seems that this is possible but we cannot do it at this moment.

- Well-posedness result for the equation (15) in the general case.

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- **Theorem 5**. (Q. Lü and X. Zhang) *Assume that H is a separable Hilbert space, and $L^p_{\mathcal{F}_T}(\Omega;\mathbb{C})$ $(1\le p<\infty)$ is a separable Banach space. Then, for any $P_T\in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$, $F\in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$ and $J,K\in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))$, the equation* (15) *admits one and only one relaxed transposition solution $\big(P(\cdot),Q^{(\cdot)},\widehat Q^{(\cdot)}\big)$. Furthermore,*

$$
\begin{aligned}
&\|P\|_{\mathcal{L}(L^2_{\mathbb{F}}(0,T;L^4(\Omega;H)),\ L^2_{\mathbb{F}}(0,T;L^{\frac43}(\Omega;H)))}+\sup_{t\in[0,T]}\big\|\big(Q^{(t)},\widehat Q^{(t)}\big)\big\|_{\mathcal{X}^2}\\
&\le C\Big[|F|_{L^1_{\mathbb{F}}(0,T;\ L^2(\Omega;\mathcal{L}(H)))}+|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\ \mathcal{L}(H))}\Big].
\end{aligned}
\tag{17}
$$

Here

$$
\begin{aligned}
\mathcal{X}\triangleq\ &\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega;H)\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)),\\
&L^2_{\mathbb{F}}(t,T;L^{\frac43}(\Omega;H)).
\end{aligned}
$$

6. Some applications of the transposition solution

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  - Study of the long time behavior of solutions to stochastic partial differential equations.

Thank you!